Sets and relations

středa 11. září 2024 10:38

2. Sets and relations

2.1. Intuitive set theory

- Set S is a collection of elements
- $x \in S$ if x is an element of S
- $x \notin S$ if x is not an element of S
- Set S is called empty, in symbols $\emptyset$, if it has no element
- Description of sets

$$S = \{1, 2, 3\}$$

$$S = \{1, 2, 3, \dots\} \quad \text{(positive integers)}$$

$$\emptyset = \{\}$$

$$\left.\begin{aligned} S &= \{x \mid P(x)\} \\ &= \{x : P(x)\} \end{aligned}\right\} \text{ set of all } x \text{ such that } P(x) \text{ holds}$$

$$S = \{x \mid x \text{ is a nonnegative real number}\} \quad (= [0, \infty))$$

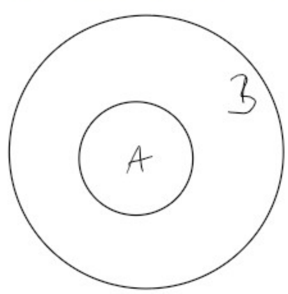
- Standard notation
  - $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of all natural numbers
  - $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ the set of all natural numbers with zero

the set of all natural numbers with zero

- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  the set of all integers
- $\mathbb{Q} = \{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$
  the set of all rational numbers
- $\mathbb{R}$ the set of all real numbers
- $\mathbb{C}$ the set of all complex numbers

Some definitions and conventions

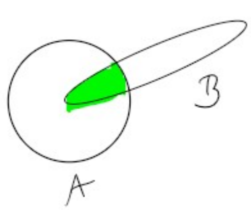
- $A = B \overset{\text{def}}{=} x \in A \Leftrightarrow x \in B$
  The sets are equal if they have same elements.
- $A \subseteq B$ A is a subset of B if $x \in A \Rightarrow x \in B$



A is a proper subset if $A \subseteq B$ and $A \neq \emptyset$, $A \neq B$.

- Intersection of sets

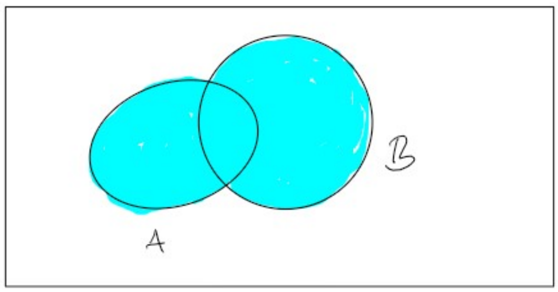
$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



If $A \cap B = \phi$ we call A and B **disjoint**

- Union

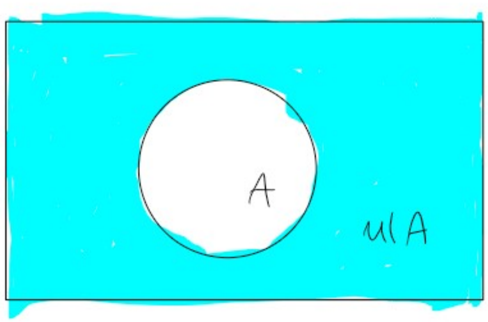
$$A \cup B = \{x \mid x \in A \vee x \in B\}$$



$\mathcal{U}$ – universal set

- Complement of A (relative to $\mathcal{U}$)

$$\mathcal{U} \setminus A = A^c = \{x \in \mathcal{U} \mid x \in A\}$$

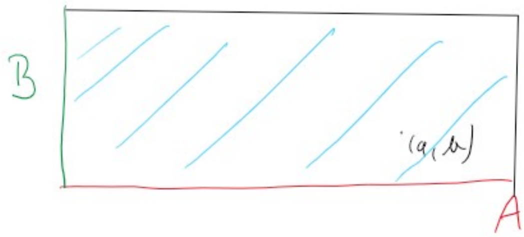


$\mathcal{U}$

- Cartesian product

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

ordered pair



Extends to $A_1 \times A_2 \times \cdots \times A_n$
$= \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n\text{-times}}$$

Examples: $\mathbb{R}^2$ - plane
$\mathbb{R}^3$ - space

- **Power set**

$$P(A) = \{B \mid B \subseteq A\}$$

Example: $A = \{1,2,3\}$

$$P(A) = \{\emptyset, A, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$$

notice that

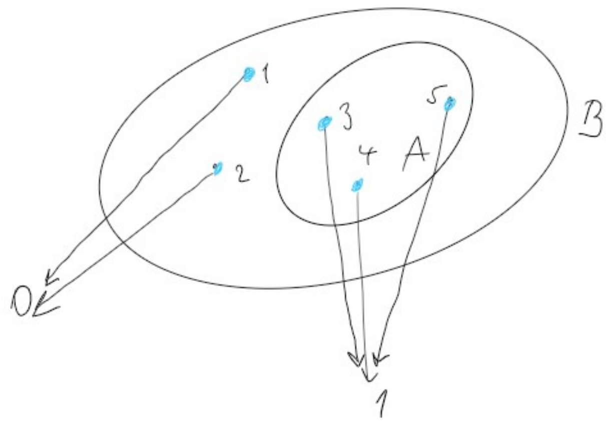
$$\boxed{\# P(A) = 2^{\# A}}$$

Hint: Encode each subset of A by a sequence of 0's and 1's of length $\# A$.

- If $A \subseteq B$ we define
  characteristic (indicator) function (relative to B)

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin B \end{cases}$$

- functions $f: A \to \{0,1\}$ $\longleftrightarrow$ subsets of B



representing sequence $(0, 0, 1, 1, 1)$

- size of cartesian product

$$\boxed{\#(A_1 \times A_2 \times \cdots \times A_n) = \#A_1 \cdot \#A_2 \cdot \cdots \#A_n}$$

Number theory

2.2. Binary relations

<u>Definition</u>: Let A and B be sets.
A (binary) relation from A to B
is the set of ordered pairs

$$R \subseteq A \times B$$

In symbols: $(a,b) \in R \equiv aRb$

<u>Example</u>: • $A = \{1, 2, 3\}$

$B = \{1, 2\}$

$R = \{(1,2), (2,2)\}$

$1R2$

$2R2$

• $A = B =$ all people

$aRb \equiv a$ is parent of b

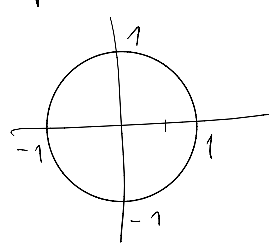
• $A = B = P(X)$

$CRD \equiv C \subseteq D$

• $A = B = \mathbb{R}$

$xRy \underset{\text{def}}{\iff} x^2 + y^2 = 1$

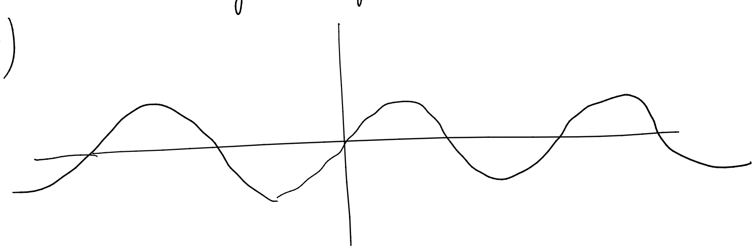
(*)



• $A = B = \mathbb{R}$

$xRy \iff y = \sin x$

(**)



Def: If $R \subseteq A \times B$ is a relation, we define its inverse

$$R^{-1} = \{(y,x) \mid (x,y) \in R\}$$

R^{-1} is a relation $\subseteq B \times A$

2.3. Functions and mappings

There is exactly one x such that $P(x)$ holds:

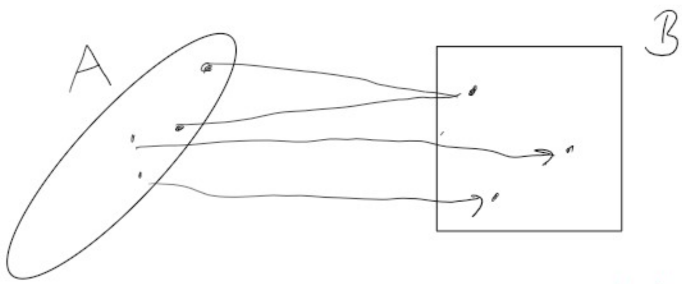
$\exists!\, x \ P(x)$

Definition a function (map) from A to B is a relation $f \subseteq A \times B$ such that

$$\forall a \in E \ \exists!\, b \in B \text{ such that } a f b.$$

In symbols: $f : A \to B$

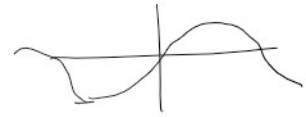
$f(a) = b \iff a f b$



Terminology: $f : A \to A$ such that $f(a) = a \ \forall a \in A$ is called identity map (function).

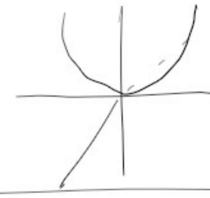
Example: • $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sin x$



• $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x, y) = x^2 + y^2$



Remark: Sometimes $f: A \to B$ is defined on a subset $D(f)$ of A, called the domain of f.

e.g. $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \sqrt{x}$

is defined on $[0, \infty)$

Notation: $f: A \to B$

$X \subseteq A$, $Y \subseteq B$

$f(X) = \{ f(x) \mid x \in A \}$ - image of X

$f^{-1}(Y) = \{ x \in X \mid f(x) = y \}$ - preimage of Y

Example: $f(x) = \sin x$

$f(\mathbb{R}) = [-1, 1]$

$f^{-1}([-1, 1]) = \mathbb{R}$

$f([0, \frac{\pi}{2}]) = [0, 1]$

Definition: a map $f: A \to B$ is called

(1) injective (one-to-one) if

$x, y \in A$ $x \neq y \Rightarrow f(x) \neq f(y)$

(2) surjective (onto) if

$\forall y \in B$ $\exists x \in A$ such that $f(x) = y$

(3) **bijective** if it is surjective and injective
(**bijection**)

Example: • $f:[0,\infty) \mapsto [0,\infty)$
$$f(x) = x^2$$
is a bijection

• $f:\mathbb{R} \to \mathbb{R}$
$$f(x) = \sin x$$
is not injective
not surjective

Some facts and observations

- $f: A \to B$ is surjective $\Leftrightarrow f(A) = B$
- $f: A \to B$ is injective $\Leftrightarrow$ $f^{-1}(\{y\})$ is a one-element set $\forall y \in f(A)$.
- $f: A \to B$ is injective if and only if f^{-1} is function
- $f: A \to B$ is a bijection $\Leftrightarrow f^{-1}: B \to A$ is a bijection

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Composition of maps

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$$g \circ f : A \to B : x \to g(f(x))$$

(Implicitly, $f(A) \subseteq$ domain of g)

- If $f: A \to A$ is a bijection then
$$f \circ f^{-1} = f^{-1} \circ f = \text{identity map}$$

- If f, g are bijections then $f \circ g$ is a bijection as well

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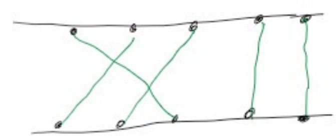
2.4 Cardinality of sets

- given two sets A and B which one is bigger?
- What is an infinite set?

Motivation: We have girls and boys

A - set of girls

B - set of boys



dancing pairs

ask shall we danse?

If no one is left then A and B has the same size.

Formalization in the following definition

Definition We say that sets A and B have the same cardinality (or are equivalent) if there is a bijection

$$f : A \to B$$

In symbols $|A| = |B|$.

We say that cardinality of A is less or equal to cardinality of B if there is an injection

$$f : A \to B.$$

In symbols $|A| \leq |B|$

We say that cardinality of A is strictly less than cardinality of B if cardinality of A is less or equal cardinality of B and A and B do not have the same cardinality.

In symbols:

$$|A| < |B|$$

Example $A = \{1, 2, 3\}$
$B = \{1, 2\}$

Then cardinality of B is strictly less than cardinality of A.

$$|B| < |A|.$$

Example $|\mathbb{N}| = |E|$

E = set of all even natural numbers

$f : \mathbb{N} \to E$
$f(n) = 2n$

$g = f^{-1} : E \to \mathbb{N}$
$g(n) = \frac{n}{2}$

Definition Let A be a set. Then A is finite if there is no bijection of A onto its proper subset.

A is infinite if it is not finite.

- $A \neq \emptyset$ is finite $\Leftrightarrow$ $|B| < |A|$ for all proper subset $B \subseteq A$

- $A \neq \emptyset$ is finite $\Longleftrightarrow$ $|B| < |A|$ for all proper subset $B \subseteq A$
- A is infinite $\Longleftrightarrow$ there is a proper subset $B \subseteq A$ such that $|A| = |B|$.

<u>Examples</u>

- $A = \{x_1, x_2, \dots, x_n\}$, $n \geq 1$ is finite
  ($\# B < n$ for every proper subset B of A)
- $\mathbb{N}$ is infinite as $|\text{even numbers}| = |\mathbb{N}|$

<u>Proposition</u> If $A \subseteq B$ and A is infinite, then B is infinite.

<u>Proof</u>: Let $C \subseteq A$ be a proper subset of A such that $|C| = |A|$.
Let $f : A \to C$ be a bijection.

Define $g : B = A \cup (B \setminus A) \to C \cup (B \setminus A)$ by formula

$$g(x) = \begin{cases} f(x) & x \in A \\ x & x \in B \setminus A \end{cases}$$

Then g is a bijection between B and its proper subset

$$C \cup (B \setminus A)$$

$\square$

Definition Let A be a set. A sequence of elements of A is a map $f: \mathbb{N} \to A$.

interpretation: $f \equiv (f(1), f(2), \dots)$

$\downarrow$ written

$(a_1, a_2, \dots) \equiv (a_n)_{n=1}^{\infty}$

$a_i \in A$

Observation: A set A is infinite $\Leftrightarrow$ there is an injective sequence of elements of A.
(i.e. $\exists\ (a_1, a_2, \dots)$ such that $a_i \neq a_j$ whenever $i \neq j$.)

Proof. $(\Rightarrow)$ A is infinite $\Rightarrow \exists\, a_1 \in A$

$\Rightarrow \exists\, a_2,\ a_2 \neq 1$ such that $a_2 \in A$ and $a_1 \neq a_2$
(otherwise $A = \{a_1\}$)

$\Rightarrow \exists\, a_3 \in A$ such that $a_3 \notin \{a_1, a_2\}$

This way we construct an injective sequence of elements of A.

$(\Leftarrow)$ Let $(a_1, a_2, \dots)$ be an injective sequence in A. Then $\{a_1, a_2, \dots, \dots\}$ is infinite and so A is infinite.

$, a_3, \dots\})$

$\square$

It says that $\mathbb{N}$ is the smallest infinite set
Such sets will be called countable

Definition: We say that a set A is countable if it has the same cardinality as $\mathbb{N}$.
An infinite sets that are not countable are called uncountable.

Observation: Infinite subset of a countable set is countable

Proof: Using suitable bijection one can suppose WLOG that $A \subseteq \mathbb{N}$, A - infinite.
Any infinite subset A of $\mathbb{N}$ is a range of increasing sequence $\{l_1, l_2, l_3, \dots\}$ of natural numbers.
Then $f: \mathbb{N} \to A$

$$f(n) = l_n \quad \forall n = 1, 2, \dots$$

Proposition: Let A be countable set and $f: A \to B$.
Then $f(A)$ is either finite or countable

Proof: WLOG $A = \mathbb{N}$

Then $f(A) = \underbrace{\{f(1), f(2), \dots\}}_{\text{either finite or countable}}$

Examples of countable sets:

- $\mathbb{N}$

- $\mathbb{N}_0$
- $\mathbb{Z}$ ... rearrange as
  $0, 1, -1, 2, -2, 3, -3$

<u>Proposition</u> Let A and B be countable sets then

- $A \cup B$ is countable
- $A \times B$ is countable

<u>Proof</u> (1) $A = \{a_1, a_2, \dots\}$

$B = \{b_1, b_2, \dots\}$

$A \cup B$ is infinite (as $A \subset A \cup B$ is infinite)

$A \cup B = \{a_1, b_1, a_2, b_2, \dots\}$ is countable

(2) $C = A \times B$ is infinite $\left(C \supseteq \{(a, b_0) \mid a \in A\} \right)$
infinite

$A = \{a_1, a_2, \dots\}$

$B = \{b_1, b_2, \dots\}$

$$C = \begin{matrix} (a_1, b_1) & (a_1, b_2) & (a_1, b_2) & \dots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \dots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \end{matrix}$$

$\rightarrow$ gives rearrangement

Example: $\mathbb{Q}$ is countable:

1. $\mathbb{Q} \supset \mathbb{Z}$ - infinite
2. $f: \mathbb{Z} \times \mathbb{Z} \setminus \{0\} \longrightarrow \mathbb{Q}$
$$(m,n) \longmapsto \frac{m}{n}$$

is a surjection.

Cantor diagonal method

The set A of all infinite sequences of 0's and 1's is uncountable

Proof: By contradiction

Suppose that A is countable and so arrange it into a sequence

$$\left.\begin{array}{llllll} s_1 & = & s_1(1) & s_1(2) & s_1(3) & \ldots \\ s_2 & = & s_2(1) & s_2(2) & s_2(3) & \ldots \\ s_3 & = & s_3(1) & s_3(2) & s_3(3) & \\ & & \vdots & & & \end{array}\right\} \text{all elements}$$

Pick up diagonal elements and create new sequence.

Denote $\neg 0 = 1$

$\neg 1 = 0$

and put

$$s = (\neg s_1(1), \neg s_2(2), \neg s_3(3), \dots)$$

Then s is sequence of 0's and 1's such that it differs from all sequences $s_1, s_2, \dots,$

This is a contradiction.

Corollary 1 $P(\mathbb{N})$ is uncountable

Proof $c \subseteq P(\mathbb{N}) \overset{1:1}{\longleftrightarrow} \chi_c \overset{1:1}{\longleftrightarrow}$ sequence of $0, 1$

Corollary 2 $\mathbb{R}$ is uncountable

Proof: Using binary expansion

CANTOR THEOREM

$$|A| < |P(A)|$$

It means that there is no "largest infinity".

2.5. Equivalence

Definition a relation R on the set A is a relation from A to A (i.e. subset of $A \times A$)

a relation on A is called

- **reflexive** if $aRa \quad \forall a \in A$
- **symmetric** if $aRb \Rightarrow bRa \quad \forall a, b \in A$
- **antisymmetric** if $aRb \wedge bRa \Rightarrow a = b \quad \forall a, b \in A$
- **transitive** if $aRb \wedge bRc \Rightarrow aRc \quad \forall a, b, c \in A$

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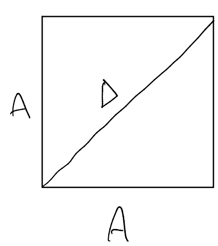
reflexivity



Δ = diagonal

$\Delta \subseteq R \Leftrightarrow R$ is reflexive

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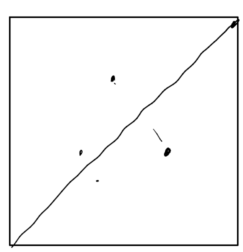
symmetry



R is symmetric $\Leftrightarrow$ it is symmetric w.r.t. Δ.

$\Leftrightarrow R = R^{-1}$

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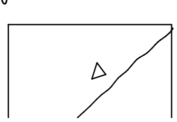
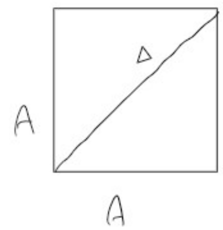
antisymmetry



antisymmetric $\Leftrightarrow R \cap R^{-1} \subseteq \Delta$

"all symmetric points coincide"



antisymmetric $\iff R^{-1} \cap R \subseteq \Delta$
"all symmetric points coincide"

<u>Definition</u> Let R be a relation on set A.
R is called **equivalence** if it is reflexive, symmetric, and transitive.

Motivating example is equality $a = b$, equality of some derived quantity like weight of people, etc.

<u>Definition</u> Let $\sim$ be an equivalence on a set A.
For every $a \in A$ we define its equivalence class

$$[a]_\sim = \{x \in A \mid x \sim a\}$$

The set of all equivalence classes is called the quotient set and denoted

$$A/\sim \; = \{[a] \mid a \in A\}$$

Examples of equivalence

- trivial equivalence on A

  $a \sim b \underset{\text{def}}{\equiv} a = b$

  $[a] = \{a\}$

- $A = \mathbb{Z}$; $n \in \mathbb{N}$

$x \sim y \underset{\text{def}}{\equiv} x-y$ is divisible by n

Later we shall talk about congruence

$x \sim y$ if, and only if,

x and y have the same remainder when divided by n, i.e.

$$x = ln + r$$

$$y = kn + r$$

$$l, k \in \mathbb{Z} \qquad r \in \{0, 1, \dots, n-1\}.$$

$\sim$ is an equivalence:

reflexivity: $a \sim a$ as $a-a=0$ is divisible by n

symmetry: $a \sim b$ then $a-b=kn$ $k \in \mathbb{Z}$

$$\Rightarrow b-a=(-k)n \Rightarrow b \sim a$$

transitivity $a \sim b$ $b \sim c \overset{?}{\Rightarrow} a \sim c$

$$\left.\begin{array}{l} b-a = kn \\ c-b = ln \\ k, l \in \mathbb{Z} \end{array}\right\} \Rightarrow \begin{array}{l} c-a = c-b+b-a \\ = ln + kn = \\ = \underbrace{(n+k)}_{\text{integer}} n \end{array} \Rightarrow a \sim c.$$

Equivalence classes $n = 2$

$[0]$ = even numbers

$[1]$ = odd numbers

$([2] = [0])$

general n ... congruence classes

$[0] = \{nk \mid k \in \mathbb{Z}\}$

$[1] = \{nk+1 \mid k \in \mathbb{Z}\}$

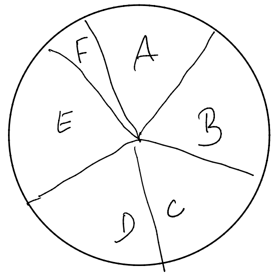
$[2] = \{nk+2 \mid k \in \mathbb{Z}\}$

$\vdots$

$[n-1] = \{nk + n - 1 \mid k \in \mathbb{Z}\}$

- A = all students in class

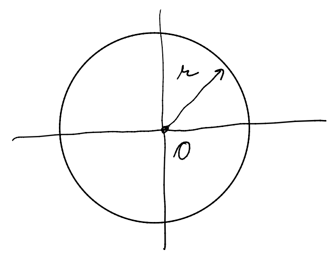
  $x \sim y$ if they have the same evaluation



- Set of all points $(x, y) \in \mathbb{R}^2$ having the same distance from zero

  Equivalence classes :

  infinitely many



- X - set

  Relation $\sim$ on $P(X)$: $A \sim B \underset{\text{df}}{\equiv} |A| = |B|$

  - It is equivalence.
  - Equivalence classes $\{\emptyset\}$, one-point subsets, two-point subset
  - X is finite $\Longleftrightarrow$ $\{X\}$ is an equivalence class

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- In all examples above we can see that equivalent classes are non-empty, disjoint, covering A. This leads to the following definition

Definition: Let A be a non-empty set.

a partition of A is a subset P of $P(A)$ such that the following conditions hold

(1) Each element of P is non-empty

(2) Two different elements of P are disjoint

(3) The union of all elements in P is the whole set A

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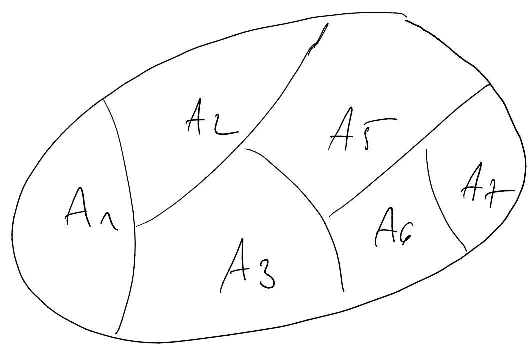
finite partition



$\forall i, j$: $A_i \neq 0$

$A_i \cap A_j = \phi$ if $i \neq j$

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_7 = A$

(*) <u>Lemma</u> Let $\sim$ be equivalence on the set A.

Then the following statements hold

(1) $a \sim b \iff [a] = [b]$

(2) a is not equivalent to b $\implies [a] \cap [b] = \emptyset$

<u>Proof</u> (1) Suppose $a \sim b$. Take $x \in [a]$. It means that $x \sim a$. But $a \sim b$ and so by transitivity of $\sim$ we have that $x \sim b$ i.e. $x \in [b]$. Therefore $[a] \subseteq [b]$.

Exchanging $a \leftrightarrow b$ we see that $[b] = [a]$ and so $[b] \subseteq [a]$. Hence $[a] = [b]$.

The reverse implication is obvious.

(2) By contradiction. Suppose $[a] \cap [b] \neq \emptyset$ and a not equivalent to b. Take $x \in [a] \cap [b]$.

Then $a \sim x$, $x \sim b \implies a \sim b$ contradiction.

Theorem: Let A be a non-empty set.

There is the following one-to-one correspondence between equivalence relations on A and partitions of A

(1) If $\sim$ is a relation, then $A/\sim$ is a partition on A

(2) If $\mathcal{P}$ is a partition of A, then the relation

$$x \sim y \underset{\text{df}}{\equiv} \exists P \in \mathcal{P} \text{ such that } x, y \in P$$

is an equivalence relation.

Proof (1)

- Each $[a] \neq \emptyset$ as $a \sim a$ and so $a \in [a]$
- By Lemma (*) $[a] \neq [b] \Rightarrow a$ is not equivalent to b

$$\Downarrow$$

$$[a] \cap [b] = \emptyset$$

- Each $a \in A$ is in $[a]$.

So $A \subseteq \bigcup_{a \in A} [a]$

(2) Let $\mathcal{P}$ be partition of A.

$x \sim y \equiv a, b \in P$ for some $P \in \mathcal{P}$.

reflexivity: As $\mathcal{P}$ is covering $\exists P \in \mathcal{P}$ $a \in P$.

Therefore $a \sim a$

symmetry: obvious

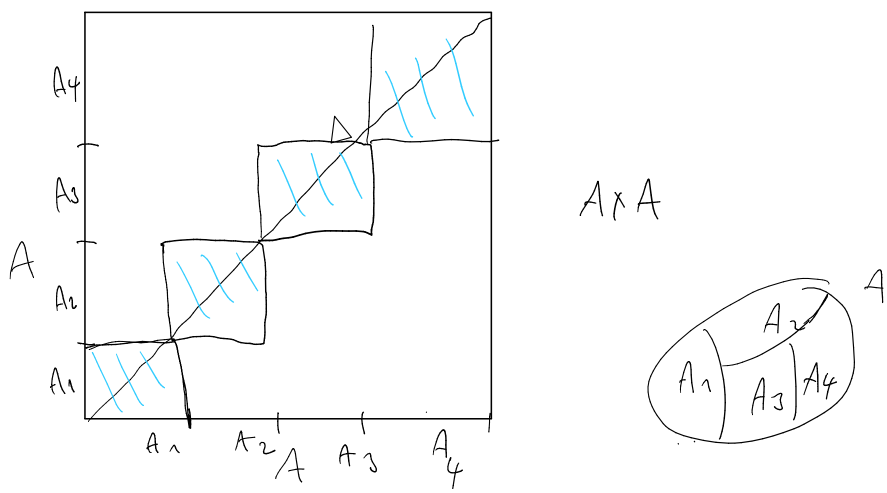
transitivity $a \sim b$ i.e. $\exists P_1 \in \mathcal{P}$ $a, b \in P_1$

$b \sim c$ i.e. $\exists P_2 \in \mathcal{P}$ $a, b \in P_2$

It implies $b \in P_1 \cap P_2$. So $P_1 \cap P_2 \neq \emptyset$.
But P is a disjoint system and so $P_1 = P_2$
which says that $a \sim c$

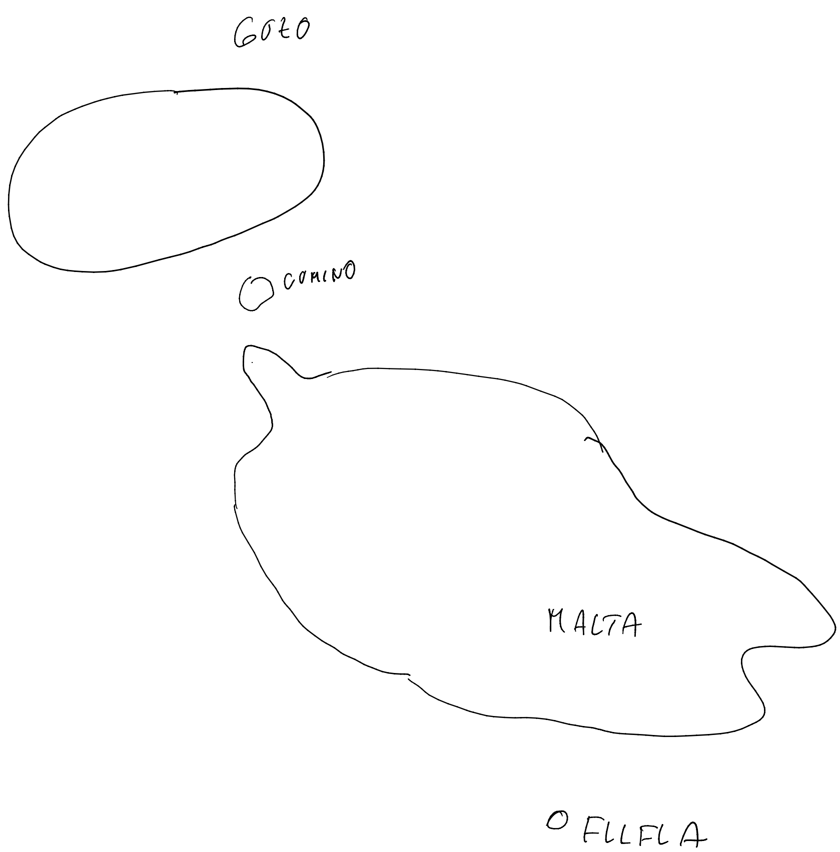
1-1 correspondence follows from the fact that
$a \sim b \iff [a] = [b]$ in Lemma (✱).

Illustration



$$R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3) \cup (A_4 \times A_4)$$

Example MALTA ARCHIPELAGO



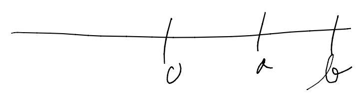
$A = \text{MALTA}$

$x, y \in A \qquad x \sim y \ \underset{\text{def}}{\equiv}$ we can get from x to y without swimming

What is $A/\sim$?

2.6. Ordering

Abstraction of $(\mathbb{R}, \leq)$:



$a \leq b$

Definition: a relation R on a set A is called partial order (ordering, order) if it is reflexive, antisymmetric and transitive.

In symbols: $aRb \equiv a \leq b$

Pair $(A, \leq)$ is called partial ordered set (POSET).

__Examples__ • $(\mathbb{R}, \leq)$ standard order

(any two elements are comparable $\equiv$ total order)

• $\mathbb{N}$ with the order $a \leq_d b \underset{\text{def}}{\equiv}$ a divides b

• $P(A)$ - power set of A. Relation $\subseteq$ (inclusion) is a partial order on A

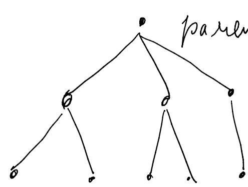
• "Family tree"

$A =$ all people

$x \leq y$ if y is ancestor of x

(we consider a person to be ancestor of herself (himself))

part of "the tree"

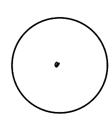


parent (generation X)

children (generation Y)

grandchildren (generation Z)

• $A = \mathbb{R}^2$ $(x,y) R (x',y')$ if $x^2+y^2 \leq x'^2+y'^2$



$\leq R$ not antisymmetric

so not order

__Definition__ Let $(P, \leq)$ be a poset and $S \subseteq P$, $x \in P$,

• we say that x is an upper bound (res. lower bound) of S if

$s \leq x \quad \forall s \in S$

(resp. $x \leq s \quad \forall s \in S$)

- If x is an upper bound of S and $x \in S$, then x is called the largest element of S.
  Analogously we define the smallest element of S.
- An element $x \in S$ is called maximal in S if $s \geq x \Rightarrow s = x \quad \forall s \in S$

  Analogously we define minimal element of S

- We say that x is a least upper bound (l.u.b) of S if the following two conditions are satisfied

  (1) $x \geq s \quad \forall s \in S$ (upper bound)

  (2) If $y \geq s \ \forall s \in S$, then $y \geq x$ (least upper bound)

terminology: l.u.b $\equiv$ supremum

In symbols

$x = \sup S$

$x = \vee S$

$x = s_1 \vee s_2 \vee \cdots \vee s_n$

if $S = \{s_1, s_2, \ldots, s_n\}$.

Typically: $S = \{a, b\}$

$\sup S = a \vee b$

- We say that x is a greatest lower bound (g.l.b) if the following holds

(1) $x \leq s \quad \forall s \in S$

(2) If $y \leq s \ \forall s \in S$, then $y \leq x$

Terminology:

g.l.b $\equiv$ infimum

$u = \inf S$

$u = s_1 \wedge s_2 \dots \wedge s_n$ if $S = \{s_1, s_2, \dots, s_n\}$

$u = a \wedge b$ if $S = \{a, b\}$.

Examples

- $(\mathbb{R}, \leq)$ $\quad S = [0,1)$

  $\sup S = 1$

  S does not have the largest element

- $P(\{1,2,3\}) \setminus \{1,2,3\}$ with set theoretic inclusion

  $\{1,2\}, \{2,3\}, \{1,3\}$ are maximal, largest element does not exist.

- $P = P(A)$ with set theoretic inclusion

  $X, Y \in P(A)$

  $X \vee Y = X \cup Y$ (union)

  $X \wedge Y = X \cap Y$ (intersection)

- P = set of all subspaces of a linear space X.

  Order is given by set theoretic inclusion. Then for $X, Y \in$ we have

  $$X \vee Y = \text{span}(X \cup Y)$$

  $$X \wedge Y = X \cap Y$$

- $\mathbb{N}$ with $\quad n \leq_d m$ if $n | m$

$\sup(n, m)$ = the smallest common multiple of n and m

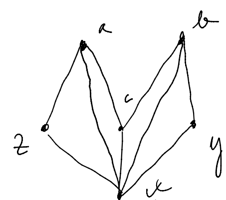
e.g. $\sup\{4, 6\} = 12$

$\inf(n, m)$ = greatest common divisor of n and m

$\inf(3, 4) = 1$

$\inf(4, 6) = 2$

- $(P, \leq)$ is given by the diagram



Then $\{a, b\}$ has upper bound, so $\sup\{a, b\}$ does not exist.

$\inf\{a, b\} = c$

<u>Definition</u> a poset $(P, \leq)$ is called a lattice if there is $a \vee b$ and $a \wedge b$ for all $a, b \in P$.

<u>Example:</u> $(\mathbb{R}, \leq)$ is a lattice